Software support for physical quantities

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Early in the 20th century, expressions involving abstract physical quantities gained acceptance. For example, Newtons third law, $F = ma$, holds regardless of units, provided we understand and apply the rules of quantity calculus. Scientists, engineers and technicians are trained to use quantity equations. Perhaps unsurprisingly, this skill is difficult to express in computer software. This paper looks at a few ideas that could be useful, some more than a century old.

1. INTRODUCTION

NASA's Mars Climate Observer crashed in September 1999 because of a serious breakdown in project management [1]. However, that accident will be best remembered for the simple error that started everything: the value of a thrust parameter was entered into computer systems in imperial, rather than SI units. That such a trivial mistake could ultimately bring about a mission failure is astonishing. It begs the question of why computers cannot help more when working with physical quantities?

Software support for physical quantities is difficult to provide. Although there have been many attempts to do so over the years, none have become established [2–17]. A survey of the literature has identified a number of aspects relating to the conceptual foundations of measurement, dimensional analysis and quantity calculus that should really be considered in designing a solution to this problem. In particular, the problem of providing a unique representation for quantities needs to be addressed, as does the classification of measurement scales and the relationship of units to both measurement scales and physical quantities. This paper provides an overview of these issues and an extensive bibliography for more detailed information.

2. THE PROBLEM

Ideally, a software framework for physical quantities would support expressions in abstract terms. For instance, a procedure for evaluating the *quantity equation* $F = ma$ might rely on a framework to take care of units and raise an error if the parameter passed as m was not a type of *mass*, or a an *acceleration*.

A popular design strategy uses a small number of quantities as a basis set and then encodes physical quantities by the set of dimensional exponents in this basis. For example, with the three quantities: length, mass and time, the force, mass and acceleration would be represented as $\{1, 1, -2\}$, $\{0, 1, 0\}$ and $\{1, 0, -2\}$ respectively. The dimensions of a product, or quotient, of two arbitrary quantities encoded in this way is easily found by summing, or subtracting, the exponents. Furthermore, with this technique the requirement that only quantities of the same type can be added or subtracted is easily checked.

This approach is quite effective. In some cases it can even be implemented without run-time overhead [14,17,18]. It has been proposed as a compact solution for peripherals [13] and adopted in an emerging IEEE standard for 'smart' sensors [19]. However, there are problems with it. The technique does not always provide a unique representation for quantities. For example, in the system above a scalar product and a vector product of directed lengths are indistinguishable: both would have a representation of $\{2, 0, 0\}$. Does this matter? Compare, for example, the work done by a force

$$W = \int \overrightarrow{F} \cdot d\overrightarrow{r} \ ,$$

which involves the scalar product and would have a representation $\{2, 1, -2\}$, with the torque associated with an applied force

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F}$$

that has the same representation but is quite a different quantity. Some notion related to the relationship of vectors defining the quantity is missing from the encoding scheme. Merely recording the dimensional exponents of a quantity can lead to ambiguity among quantities.

Another problem arises from the semantics of the representation. Is it meaningful, for example, that a specification of paint coverage in square meters per litre reduces to the dimensions of inverse length? Similarly the common unit for petrol consumption, litres per 100 km, cannot be distinguished from surface area! Abstract support for physical quantities will have to deal with such cases, so a way of maintaining better semantic information is needed. This is especially true for ratios of a physical quantity, which have the same representation as pure numbers.

A further weakness of the approach is that it often ignores the type of measurement scale. It is usually assumed that a change of units can be effected by an appropriate scale factor, but this is not always true. A familiar counter example being the conversion of temperature data between Fahrenheit and Celsius.

3. MEASUREMENT SCALES

A measurement associates a numerical value with a particular physical quantity. However, the values assigned should not be endowed with all the properties of real numbers. It is possible to classify measurement procedures according to the type of information they can provide about a quantity. Stevens has identified four classes of measurement scale: nominal, ordinal, interval and ratio [20]. A *nominal* scale is obtained when measurement is capable only of determining equivalence; an *ordinal* scale can also determine relative order (i.e. greater than and less than). These scales are rarely important in physical measurements* but it seems quite reasonable to support them in general purpose software. For example, exam scores arguably belong to an ordinal scale. An *interval* scale is obtained when a measurement procedure can determine equal intervals, such as some temperature scales and time. The zero point on an interval scale is chosen by convention, or for convenience, as in the case of the Celsius and Fahrenheit scales.† Measurements on a *ratio* scale require a procedure that can determine the equality of ratios, which implies a common zero for all scales. Most physical quantities can be measured on ratio scales, which is why it is commonly assumed that measured values can be multiplied by a scale factor when changing units.

The type of scale determines the operations appropriate for distinguishing between measurement values, which for example restricts the kind of statistical analysis applicable to a set of measurement data. For instance, it is not strictly correct to calculate the mean and variance of a sample of ordinal data, because the extent of differences between data is not known; order statistics, like the median and percentile, are appropriate. It is also worth noting that logarithmic transformations (e.g. decibels) should only be used with ratio scale data.

*Scales of mineral hardness are an example of ordinal scales.

†Absolute temperature scales, such as the kelvin and Rankine scales define a common absolute zero and can therefore be considered ratio scales.

4. UNITS, QUANTITIES AND DIMENSIONS

It is convenient to identify a small set of so-called fundamental units, which are deemed sufficient for measuring the properties of a class of phenomena of interest (e.g. mechanics). The set is called a *system of units*, for example: the CGS system (centimetre-gram-second), the MKS system (metre-kilogram-second) and the SI system (Système Internationale). The choice of fundamental units is to a certain extent arbitrary and the number of fundamental units a matter of discretion.

Derived units are defined by describing a measurement of the quantity. Velocity, for example, is the ratio of a distance to the time interval taken to travel that distance. The unit of velocity is such that, under conditions of uniform motion, one unit length is covered in one unit of time. Note that there is no mention of dividing quantities, although the definition implies taking the ratio of the numerical measures of the quantities involved.

Different systems of units can use the same set of fundamental quantities, in which case they are said to belong to the same *class of systems of units* [22], or *quantity system* [23]. Symbols for the physical quantities designated as fundamental, such as LMT, may be sufficient to identify a quantity system. However, it is also possible that two systems of units use the same fundamental quantities yet define a different set of derived quantities, as in the old electromagnetic and electrostatic unit systems [23].

The concept of a quantity system is useful in understanding unit conversions. When one system of units is changed to another in the same class, the accompanying change to the numerical value of a quantity depends on its dimensional exponent. For example, if a force is measured in units of (g cm s^{-2}) and a change of units to (kg m h^{-2}) is required, the fact that dimensional exponent of both mass and length is unity, and of time is minus two, leads to a conversion factor of $129.6 = \frac{1}{1000}\frac{1}{100}\frac{3600^2}{1}$. Physical dimensions are often associated with systems of units. The notion of dimensions was introduced by Fourier to account for the conversion factors required in expressions involving physical quantities when the units of measurement were changed [21]. Nevertheless, it is important to recognize that this original use of dimensions applies only to measurements on a ratio scale. As a special case, the value of a *dimensionless* quantity remains the same in all systems of units of a particular class.

Note, that the dimensions of a quantity may change in a different class of system of units. For instance, force, length and time (LFT) provide an alternative quantity system for mechanics, in which the dimensions of mass are $[m] = L^{-1}FT^2$. Therefore, the conversion of units belonging to different classes of systems of units must consider the relationships between quantities.

5. QUANTITY CALCULUS

The notion of physical quantity was introduced by Maxwell to conveniently express abstract physical relationships [23]. For example,

$$v = \frac{dx}{dt} \ , \ E_\mathrm{k} = \frac{1}{2}mv^2 \ ,$$

define derived quantities without reference to a choice of units, similarly,

$$F = ma \ , \ E = mc^2 \ ,$$

describe physical laws.

A physical quantity has two components: a value and a measurement unit[‡]. Symbolically, a quantity Q can be represented as

$$Q = \{Q\} \cdot [Q] \ ,$$

where $\{Q\}$ is the numeric value and $[Q]$ is the unit. Rules for algebraic operations need to be developed for the abstraction. Addition and subtraction require that quantities have the same unit. In multiplication (division is the inverse of multiplication [24]), unit and value components are handled separately. The product $C = A \cdot B$ can be expanded as

$$C = A \cdot B = \{A\}[A] \cdot \{B\}[B] \ . \qquad (1)$$

The units of C are related to the units of A and B by a numerical proportionally factor α (e.g. an acre is 4840 square yards or $\frac{1}{640}$ of a square mile). So[§]

$$[C] = \alpha[A][B] \qquad (2)$$

and the value of the product is (after substituting for $[C]$ in (1) and cancelling $[A] \cdot [B]$ on both sides of the equation)

$$\{C\} = \frac{1}{\alpha}\{A\}\{B\} \ . \qquad (3)$$

6. HIDDEN DIMENSIONS

The difficulty of distinguishing between scalar and vector products mentioned earlier is well known. Lodge referred to it in 1888 and made the comment regarding the role of *angle* in these products "Again, the circular measure of angle is not a pure number, though it is of zero dimensions as a pure number is ..." [24]. He went on to outline some heuristic *laws* applying to expressions involving directed quantities *for which the symbols themselves do not include the idea of direction*:

- if any term is independent of direction, then all terms must be, or involve, ratios between *parallel* vectors;
- if any term involves a vector, all other terms must involve the *same* vector (after simplification);
- if any term involves a product (or ratio) between two vectors including an angle, then all terms (after simplification) must involve a product (or ratio) between vectors involving the same angle.

Consider the work done by a force F applied to a bar of length r and turning through an angle θ. Angle is the ratio of an arc length to a radius (perpendicular directions) so $r\theta$ simplifies to an arc (second rule) and $Fr\theta$ is independent of direction (first rule). The third rule might apply, for example, to distinguish an area from the product of two parallel lengths.

A few years later, Williams suggested using an extension of dimensional formulae to identify various physical quantities [25]. He regarded the key issue as "...the manner in which the unit of a quantity is built up from fundamental units and not merely the manner its magnitude *changes* with those units." Williams sought to extend the semantics of dimensional formulae so that the physical nature of any quantity could be identified symbolically, which is the very problem a software representation faces. He considered lengths along three perpendicular directions of space as being distinct and, instead of L, used X, Y and Z to label these quantities. Thus the dimensions for *area* can be any of XY, YZ or ZX; *volume* XYZ; *angle* any of XY^{-1}, YZ^{-1} or ZX^{-1} and so on.

‡Note that the quantity calculus applies only to measurements belonging to ratio scales.

§In a coherent system of units the factor α is always unity, so numerical equations have exactly the same form as quantity equations.

The extra dimensions seem to resolve the basic ambiguities (although no claim of completeness is made). In practice, equations will need to be expressed in three dimensions. Williams' method is then applied to the terms of the Cartesian coordinates, i.e. if A, B, $\cdots$, are quantities related by the equation

$$A + B + \cdots = 0 \tag{4}$$

then, in terms of the x, y and z coordinates,

$$(A_x + A_y + A_z) + (B_x + B_y + B_z) + \cdots = 0 \tag{5}$$

and the dimensions of every term can be assigned. The dimensional homogeneity of such equations should apply to terms along the same Cartesian axis, i.e. the terms $(A_x + B_x + \cdots)$ will have the same dimensions, and therefore represent identical physical quantities, similarly $(A_y + B_y + \cdots)$ and $(A_z + B_z + \cdots)$. The components of quantities along different axes will be of the same kind but may differ in orientation.

Discrimination between coordinate directions has been recognised by others as a useful technique. Huntley provides a number of examples using it [26]** and Krantz also mentions the approach [27]. These writers recognize, too, the unusual nature of 'angle' and the need to assign a dimension to it, however they do not adopt Williams' ratio of perpendicular length dimensions.

Quite a lot has been written on whether or not *angle* should be treated as a distinct quantity or simply a dimensionless pure number [28–36]. In the SI, plane angle is a dimensionless ratio of two lengths [37]. However, the fact remains that in the *dimensional analysis* of specific cases it is often useful to treat angle as a distinct dimension of the problem (albeit a dimensionless quantity). Finding a way of encoding the dimension of angle will be an important aspect of any software framework and there does not yet appear to be a conventional solution available. A suggestion by Page, that the dimension of angle have the property $[\theta]^2 = 1$, is potentially useful [28,30]. This comes close to Williams' comment that the versors i, j, k might be used in conjunction with the dimension L to designate different directions in space (i.e. $iL \equiv X$, etc) [25]. It is also related to ideas expressed by Cahen on an axial view of dimensional analysis [38]. Unfortunately, the idea is rather at odds with the familiar notions of angle. It could be controversial if incorporated in a design.

**Huntley also suggests that the *mass* dimension can be usefully resolved into two distinct quantities in some thermal problems [26].

7. DIMENSIONAL VECTORS AND MATRICES

Computations involving physical quantities are frequently cast into vector and matrix forms†† both for convenience of expression and for efficiency and ease of calculation. In such cases, the vector and matrix elements are treated as pure numbers so the various restrictions that might apply to manipulation of actual physical quantities are again ignored.

Hart has made an extensive study of dimensional linear algebra [39]. The most fundamental difficulty that arises when physical quantities are manipulated using linear algebra is the constraint imposed by addition and subtraction of quantities. For example, the conventional dot product of two *dimensional* vectors may not be defined because the element-wise sum of products may not be compatible. A special case of this implies that the squared magnitude of a vector can only exist if all elements of the vector are of the same type.

Dimensional matrices are restricted with regard to the operations that can be performed on them. For example, the matrix product, $\mathbf{AB}$, is only defined when certain conditions on $\mathbf{A}$ and $\mathbf{B}$ are met. A matrix product is actually an array of dot products between the rows of one matrix and the columns of the other. It turns out that for $\mathbf{A}$ to be a *multipliable* matrix‡‡ there must exist a pair of vectors $\mathbf{a}$ and $\mathbf{b}$ such that the elements of the outer product $\mathbf{ab}^{\mathrm{T}}$ have the same dimensions as the elements of $\mathbf{A}$ (similarly, there must exist a pair of vectors such that $\mathbf{cd}^{\mathrm{T}}$ matches the dimensions of $\mathbf{B}$). The product $\mathbf{AB}$ will then only be defined if certain conditions are met between the vectors $\mathbf{b}$ and $\mathbf{c}$. An intuitive view of this can be gained by considering the transformation of a dimensioned vector $\mathbf{x}$ into a vector $\mathbf{y}$ belonging to another dimensioned space

$$\mathbf{y} = \mathbf{Ax} \; . \tag{6}$$

Multiplying by $\mathbf{A}$ on the left must in a sense do two things to transform the dimensions of one space to the other: cancel the dimensions of the elements of $\mathbf{x}$, by multiplying with inverse dimensions, and set the dimensions of $\mathbf{y}$. Hence a dimensional vector that is the inverse of $\mathbf{x}$ and a vector with the same dimensions as $\mathbf{y}$ characterise the dimensions of matrix $\mathbf{A}$.

††Here we are considering the mathematical construct of a vector as an arbitrary grouping of n elements – an n-tuple, as opposed to a directed physical quantity such as velocity or force. In this sense a vector may be composed of elements representing diverse physical quantities. For instance, a vector might be used to record the state of an electrical circuit at some point using elements of voltage and current.

‡‡i.e. that the matrix be eligible for multiplication at all

An important aspect of Hart's work is his claim that *only* multipliable dimensional matrices are in fact useful. Although this is conjecture, it allows considerable saving of resources in encoding and implementing matrix algorithms for dimensional matrices. Only two n-vectors need be stored and manipulated for any $n \times n$ matrix of dimensioned elements. The dimensional aspect of any matrix algorithm can be handled efficiently and indeed separately from the numerical part of the calculation, which permits the use of standard numerical libraries for linear algebra.

Hart describes a rich mathematical structure implicit in dimensional matrices. In addition to the limits placed on matrices that may be meaningfully multiplied with other matrices, only some matrices will poses an eigenstructure, or have a determinant or an inverse. Nevertheless, some of Hart's assumptions regarding dimensioned quantities are questionable. In particular, his rejection of the order operations '$<$' and '$>$' seems incompatible with the most fundamental aspects of measurement theory! Similarly, the arguments for disallowing the modulus and square root of a dimensioned quantity seems unwise.

8. DISCUSSION

This work set out to identify the key conceptual elements of physical quantities so that they can be included in the design of software frameworks. The previous sections provide an overview of our findings in the literature. It appears, that many software implementations have overestimated the power of dimensional exponents to identify physical quantities. The dimensional homogeneity trick, taught to all physical science and engineering students as a way of checking equations, is only a necessary condition for correctness, not a sufficient one. This does not seem to be widely understood. A more robust approach should acknowledge the importance of measurement scales and take advantage of opportunities to build classification hierarchies based on classical dimensional analysis.

In designing a framework, relationships between physical quantities should be expressed as much as possible in abstract terms. This follows the approach of Maxwell's quantity calculus, but must be broader so as to admit different types of measurement scale. Abstract quantities, such as length, voltage, etc, must be quite distinct from the associated measurement units (metre, volt, etc). Instances of quantities assume the concrete attributes of value and units.

Where systems of units are employed, the notion of quantity system will be important because it underlies conversions between units. Nevertheless, there is problem in uniquely identifying physical quantities if only the fundamental quantities in a particular systems of units (such as the SI) are used as a basis. One approach may be to follow classical dimensional analysis and select a basis of quantities relevant to a particular problem. For instance, natural quantities in a problem involving fuel consumption could include volume and distance traveled – it is better not to reduce these both in terms of length. A similar strategy should be adopted in software design. However, it will be difficult to strike a balance between correctness, flexibility and the desire to support, rather than overburden, the users of a framework.

Measurement scales are a fundamental concept. It is important to recognize that measurement *units* and measurement *scales* are closely related. Most implementations have directly associated quantities with ratio scales, which is incorrect. It is possible to measure the same quantity with units belonging to different types of scale (temperature is the obvious example).

Scales describe the inherent limits on information contained in a set of measurement data. There are many familiar instances of interval scale data: spatial location, the date and temperature, for example. With this in mind, some arithmetic operations are clearly inappropriate: the sum of two values on an interval scale, for example, is meaningless although the difference is valid and produces in a value belonging to a ratio scale. Simple rules can be formulated governing the arithmetic of measurement scales. If **I** represent values on an interval value and **R** values on a ratio scale, then some of the rules are

- $\mathbf{I} + \mathbf{I} \rightarrow$ not defined
- $\mathbf{I} - \mathbf{I} \rightarrow \mathbf{R}$
- $\mathbf{I} + \mathbf{R} \rightarrow \mathbf{I}$ (and $\mathbf{R} + \mathbf{I}$)
- $\mathbf{I} - \mathbf{R} \rightarrow \mathbf{I}$ (and $\mathbf{R} - \mathbf{I}$)
- $\mathbf{R} + \mathbf{R} \rightarrow \mathbf{R}$
- $\mathbf{R} - \mathbf{R} \rightarrow \mathbf{R}$

In conclusion, there is quite an extensive domain of knowledge that should be considered when designing a framework to support physical quantities as a software abstraction. Designs to date have not recognised this and as a consequence have run into difficulties. A general lack of acceptance for proposed schemes is almost certainly exacerbated by the habitual use of pure numbers to represent physical quantities in software – a practice that is widely recognised as error-prone.

Based on the ideas outlined above, it seems quite feasible to create robust support for physical quantities in software. However, there will be several challenges in doing so. Firstly, any scheme will be closely scrutinized with regard to efficiency; a significant burden on memory resources or execution time must be avoided. Secondly, the conviviality of the framework – the user interface – will be immensely important. If the framework is not easy to use correctly it will not be used at all.

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